“TOY” DYNAMO TO DESCRIBE THE LONG-TERM SOLAR ACTIVITY CYCLES

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Abstract. Secular variations of solar activity (Gleissberg and Suess cycles) have approximately 80-130 and 200 year periods. They are manifested in both observed and proxy data. Here we show that the basic dynamic features of the Schwabe cycle (asymmetry of its growth and decay phases) and secular cycles (multi-frequency structure and irregular Grand-extremes) as well as a connection between them can be described by parameter tuning of the electro-mechanical “toy” dynamo system which was widely used to model the inversions of the geomagnetic field. An amplitude-frequency diagram for the model magnetic flux has the same shape as the directly observed and reconstructed sunspot area indices.

Solar cycles, disc dynamo, paleo-reconstructions

1. Introduction

Hale’s magnetic cycle is one of the well-known phenomena in solar physics but a number of peculiarities of its temporal behavior still need more adequate description and understanding. Evidence for the presence of its long-term modulation and occurrence of epochs of reduced activity (referred to as grand minima) comes from a variety of sources, including studies of the historical records of observed sunspots since 1607 AD (Hoyt and Schatten, 1996). Further research and modeling of the solar activity based on historic records of observed aurorae (Nagovitsyn, 1997), cosmogenic ^{10}Be isotopes in polar ice cores (Bard *et al.*, 2000 and references therein) and ^{14}C in tree rings (Solanki *et al.*, 2004 and references therein), and archaeomagnetic data dispersion (Volobuev, 2004) provided additional strong evidence that the grand minima have persisted with irregular intervals in the past.

The irregular occurrences of grand minima in the sunspot activity are of interest for at least two reasons. First is that they are theoretically challenging, especially in view of difficulties to find any naturally occurring mechanisms in the Sun with appropriate time scales (Solanki and Schüssler, 2005). Secondly, these variations can be of potential consequence for the occurrence of climatic variability on the similar time scales (*e.g.* Ogurtsov *et al.* 2002).

Typical magneto-hydrodynamic (MHD) dynamos are governed by the nonlinear partial differential equations (PDEs) to be solved under realistic boundary conditions (see *e.g.* Durney, 2000). More promising for clarifying temporal behavior are 'intermediate' models of MHD dynamos in which the nonlinear PDEs are simplified by carrying out spatial averages (*e.g.* Kitchatinov *et al.*, 1999). The low dimensional ordinary differential equation (ODE) models that are obtained using the Normal Form approach (*e.g.* Knobloch and Landsberg. 1996) also is of potential importance in accounting for certain aspects of solar variability, such as an amplitude modulation of the magnetic field.

An alternative way to understand grand minima in the sunspot record is to postulate that this type of behavior is caused by some form of the dynamical intermittency. This idea in various forms goes back to the late 1970s (*e.g.* Tavakol, 1978, Ruzmaikin,1981).

A simpler way to investigate the temporal behavior of self-exciting dynamos is exploited in the present paper. It is based on analyses of the non-linear ODEs in the single independent variable, time, which govern the behavior of the electro-mechanical system for coupled Faraday discs. The pioneering study by Rikitake (1958) helped to understand the origin of the chaotic sequence of geomagnetic inversions which was modeled by the system of two ideal (non-dissipative) magnetically connected discs. This system is described by three first-order ODEs only. Ershov, Malinetskii and Ruzmaikin (1989) undertook a study of the generalized disc model. Hide (1997) extended the model and included both dissipation and energy depositor in the system, which consisted of one or several magnetically connected discs. Apart from the undoubted mathematical interest of the solutions of the governing ODEs, this approach treats physically realistic systems and provides general insights into the likely behavior of the more complex MHD systems.

In the present work we use this approach to describe a few of the most interesting dynamical peculiarities of long-term solar activity cycles, including:

- Growth of Schwabe cycle is shorter than its decay phase.
- Smaller cycles are longer.
- Positive Lyapunov exponent (predictability horizon) exists in the evolution of long-term variations of solar activity.
- The amplitude – frequency diagram has a well-defined shape.

We investigate the systems of ODEs by Rikitake and Hide with different parameters and initial conditions in order to fit the Schwabe cycle and the Gleissberg cycle as they appear in the sunspot area indices.

2. The model of the Hale cycle

The ODEs investigated by Rikitake (1958) describe electromechanical oscillations in the system of two rotating conductive discs coupled with the electromagnetic inductance (modified scheme is shown in Figure 1, upper panel). Initial rotation and magnetic field exist in the system, mechanical friction is absent. Both disc-coil systems have the same mutual (*M*) and self (*L*) inductance, resistance (*R*), inertia (*C*) and are acted on by the same constant torques (*N*). The equations of motion are:

$$\begin{aligned}\dot{X} &= -\mu X + ZY \\ \dot{Y} &= -\mu Y + (Z - A)X \\ \dot{Z} &= F_0 - XY\end{aligned} \qquad (1)$$

In these, the constants are given by $\mu = (C/NLM)^{1/2} R$ and $A = Z - V = \mu(K^2 - K^{-2})$. F_o is defined here as the turning parameter, in standard system with equal torque of the discs F_o =1.

The variables are given by:

$X, Y = (M/N)^{1/2} I_{1,2}$, - the current through the disc 1 and 2 respectively, $Z, V = (CM/NL)^{1/2} \omega_{1,2}$, - the rotation (angular velocity) of discs. Equations (1) describe the transformation of the mechanical rotation energy into the energy of magnetic field (current) and the reverse. A standard parameter set $\{\mu=1, K=2, F_o=1\}$ produces a typical solution, which describes sudden (chaotic) geomagnetic inversions. ODEs (1) were derived under the assumption that only disc-coil interaction is effective (no disc-disc and coil-coil interactions). In this case the orientation of the disc axle has no effect on the ODEs (1). When axles of the discs are not parallel as it is shown in Figure 1, one disk may represent deviations from the average rotation of the Sun and another one may represent its meridional convective fluxes (Figure 1, right panel).

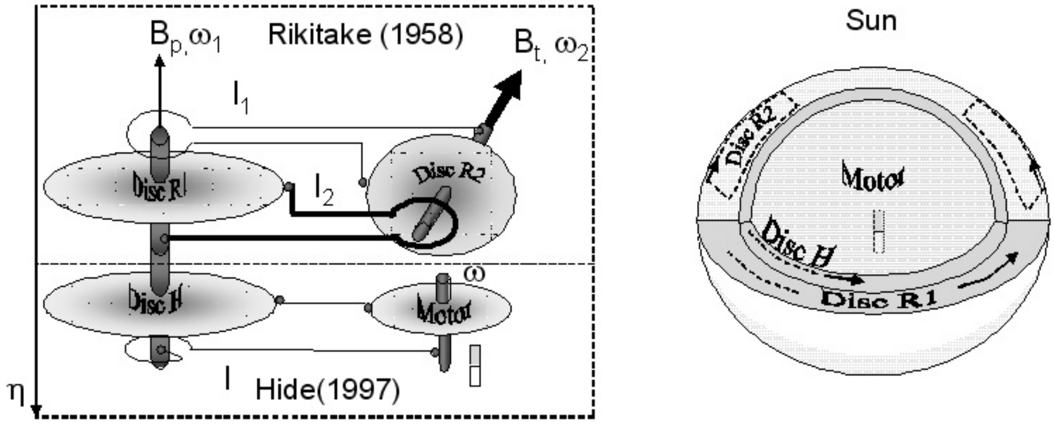

Figure 1. The geometry of electro-mechanical dynamo systems and its possible relations to the Sun. Upper panel: Two-disc (Rikitake,1958) dynamo described by ODEs (1), direction of the disc R2 is changed to complete the analogy with the meridional convection. Each disc rotates in the magnetic field which is produced by the induced current in the other disc. The coil of the I_2 -current produces the "toroidal" magnetic field B_t in parallel with rotation of first disc ω_1. In a similar way, the current I_1 produces the "poloidal" field B_p.

Bottom panel: One-disc (Hide, 1997) dynamo with "motor" and mechanical friction described by ODEs (2). Constant torque is applied to rotate Disc H. The permanent magnet of the "motor" induces the current through its disc. The viscosity (η) of the underlying MHD system is higher, discs are rotated slower and the mechanical friction is not negligible.

Both dynamo systems are described by ODEs (3) when they are connected through the sliding mechanical contact between Disc H and Disc R1.

It is useful to consider a coordinate system which is rotating with the Sun. This makes the torque of the discs approximately equal and preserves the general symmetry of ODEs (1) but causes the coriolis force to appear in the last equation. We set the parameter F_o to be essentially negative to take this effect into account at first approximation. This leads to significant changes in the topology of the phase-space (Figure 3, A, B). The parameter set $\{\mu=1, K=2, F_o=-11\}$ produces a periodic solution (Figure 2, Figure 3B) with several features of the Hale solar cycle. The current I_2 of the second disc has the shorter growth phase and longer decay phase with the tendency to double its maxima, which is the typical behavior of most indices of solar activity (sunspot numbers, sunspot areas, open magnetic flux).

More accurate comparison of the model output and actual sunspot indices can be achieved by considering the value $ME_t = \frac{1}{8\pi}\int_V <B_t>^2 dV \sim I_2^2$ that is the energy of the magnetic field produced by the current I_2 in the volume V. The squared current I_2 is supposed to be responsible for the energy of the toroidal magnetic field of the Schwabe cycle. The disc-coil system exhibits a “differential” rotation. That is, different parts of the conductor are rotating with different angular velocity (the discs with ω_1, the wire not rotating). Changes of the rotation of the Disc R1 (Figure 1) could be interpreted as deviations of the solar differential rotation relative to its average profile. These deviations are due to the energy transfer from the rotational energy into magnetic energy and could be interpreted as the global torsional oscillations discovered by Howard and LaBonte (1980). Following Yoshimura (1981), we believe that these oscillations are generated by the Lorentz force associated with the solar dynamo.

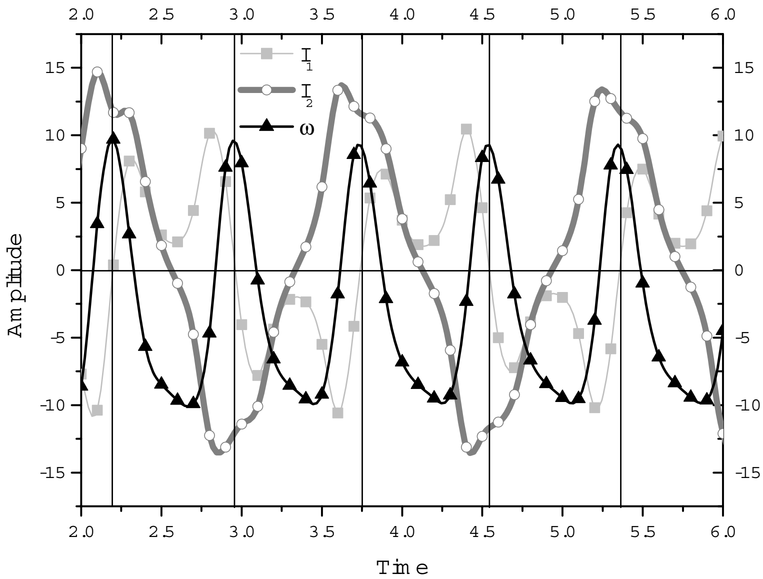

Figure 2. Periodic solution in the Rikitake (1958) system of ODEs with the adopted parameters. Time and amplitude are expressed in arbitrary units. Sampling relative to the 22-year period corresponds approximately to that of annual sunspot indices (ten points per half-period). The ODEs are solved with the standard Runge-Kutta fourth-order scheme.

I_1 and I_2 - “toy” dynamo currents are interpreted here as responsible for the poloidal and toroidal magnetic fields respectively. ω - deviation from the average rotation. Vertical grid lines correspond to the coinciding maxima of ω and changes of the sign of I_1.

With available contemporary solar activity indices we cannot find an adequate interpretation of the model current I_1. Most likely, this current may be responsible for the poloidal magnetic field. It has smaller amplitude than the current I_1 and changes its sign after extremes of the current I_2 as the polar magnetic field does, but the broad gap in the poloidal magnetic field during the minima of the Schwabe cycle is difficult to find in the solar indices. Certainly, the proposed interpretations should be referred to as “toy” interpretations.

3. The model of secular variations

Hide (1997) found that the Rikitake system is not realistic because its “discs” rotate without “friction”. He introduce an energy source (“motor”) and dissipation (“friction”) into the model (Figure 1, bottom panel). Denote by M the mutual inductance between the disc and coil; G the steady torque applied to the disk, $\{D, B\}$ and $\{K, A\}$ the friction constant and the moments of inertia for the motor and disc respectively, H the magnetic inductance of the motor’s permanent magnet, R the total electrical resistance, and L the total self-inductance Hide (1997) found the following set of equations for his model:

$$\begin{aligned} \dot{x} &= x(y-1) - \beta z, \\ \dot{y} &= \alpha(1-x^2) - \kappa y, \qquad (2) \\ \dot{z} &= x - \lambda z. \end{aligned}$$

Here $x(\tau) = (M/G)^{1/2} I(t)$ is the current,

$y(\tau) = (M/R)^{1/2}\Omega(t)$ is the angular velocity of “disc” rotation, and

$z(\tau) = (M/G)^{1/2}(RB/LH)\omega(t)$ is the angular velocity of “motor” rotation.

Non-negative dimensionless parameters of ODEs (2) are defined as: $\alpha = GLM/R^2 A, \beta = H^2 L/R^2 B, \kappa = KL/RA$ and $\lambda = DL/RB$. The dependence of ODEs (2) on parameters $\{\alpha, \beta, \lambda, \kappa\}$ was studied numerically and analytically by Hide and Moroz (1999). In the absence of the motor, the parameters β and λ are both equal to zero, and when κ also is equal to zero, Equations (2) describe the case of the famous homopolar dynamo discussed by Bullard (1955). Dynamo action occurs when effects due to motional induction suffice to overcome the dissipation:

$$\frac{\alpha}{\kappa} > \min\left\{1 + \frac{\beta}{\lambda},\ 1+\lambda\right\} \qquad (2a)$$

Considering the Sun, inertia A of the main disc is much more significant than mechanical friction, and the constant κ can be equal to zero. In this case, condition (2a) is always true. Steady-state solutions are: $x = \pm 1; y = \frac{\beta}{\lambda} \pm 1; z = \pm 1$.

In the dynamic case, these points are unstable focii with phase trajectory moving around them and producing periodic or chaotic solutions similar to these of the ODEs (1) (Figure 3, A and B).

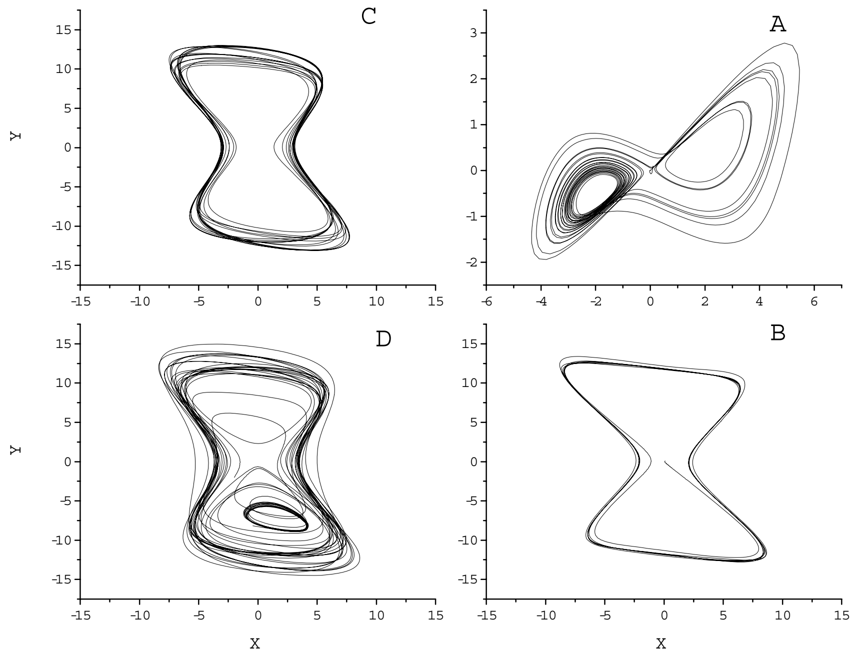

Figure 3. Solution of ODEs in phase space.

A : Rikitake (1958) system with standard parameters. A chaotic trajectory can move around each of positive and negative focii or both of them. B: Rikitake (1958) system with F_0= -11, periodic trajectory attracts to the limit-cycle and moves around both focii. C: combined system (3) with weak chaotic modulation (ε=1.6), time dependent curve is shown in Figure 4, upper panel. D: combined system (3) with strong chaotic modulation (ε=3.2). Chaotic solution behaves like case (A).

The parameter set $\{\alpha=5, \beta=4, \lambda=1.2, \kappa=0\}$ of system (2) can produce a solution with variable “y” which mimics the shape of the long-term variation of solar activity (Figure 4, upper panel, envelope). The chaotic trajectory can move around each of the positive and negative focii or both of them similar to that of system (1) (Figure 3A).

Varying of each parameter $\alpha \in [6, 4.5]$ or $\beta \in [3, 4.5]$ or $\lambda \in [0.8, 1.9]$ or $\kappa \in [0, 0.2]$ maintains this type of behavior. Solutions tends to be periodic (attracts to the limit-cycle in the phase space) or stochastic (attracts to the torus) in the case when $\{\alpha, \beta, \lambda\}$ are outside of these ranges and κ remains small enough. Solutions of this type are too simple to describe the behavior of the long-term solar cycles. The parameter set $\{\alpha=5,\ \beta=4,\ \lambda=1.2,\ \kappa=0\}$ of this system can produce the solution with variable “y” which mimics the shape of the long-term variation of solar activity (Figure 4, upper panel, envelope). The inverse Lyapunov exponent (the horizon of predictability) of the model estimated with the algorithm by Wolf *et al.* (1985) is found to be 210 years, which is the approximate length of the Suess cycle. Unlike the Hale cycle described by the variables of equations (1), it is much more difficult to speculate about the physics underlying variables of equations (2) – it would be better to consider them as being responsible for the hypothetical MHD variations of the current and rotation on the century time scale. However, the motor’s permanent magnet could be associated with the hypotheses of the internal fossil magnetic field and Disc H with the tachocline (Figure 1).

Following Kitchatinov et al. (1999) we suppose that the mechanism of the modulation of the Hale cycle is due to long-term variations in solar rotation. In terms of the models presented, it means that variable “y” of the system (2) should be included in the “rotational” equation for the variable “Z” of system (1). The combined system of ODEs becomes:

$$\begin{aligned}
\dot{X} &= dt(-\mu X + ZY) \\
\dot{Y} &= dt(-\mu Y + (Z - A)X) \\
\dot{x} &= d\tau(x(y-1) - \beta z) \\
\dot{y} &= d\tau\alpha(1 - x^2) \\
\dot{z} &= d\tau(x - \lambda z) \\
\dot{Z} + \varepsilon\dot{y} &= dt(F_o - XY),
\end{aligned} \qquad (3)$$

where $dt = 1.6$ and $d\tau = 0.2$ are coefficients of the time scaling. ε - transmission coefficient of the rotation from Disc H to Disc R1 (Figure 1). The system of ODEs (3) was solved with the standard Runge-Kutta fourth-order scheme. The calculated magnetic energy is defined as the sum of squared currents through each disc:

$$ME \equiv X^2 + Y^2 + x^2$$

It is shown in Figure 4, upper panel. The shape of “11-year”cycles of ME is defined mostly by the variable “Y”, which is supposed to be responsible for the magnetic energy of the toroidal field, whereas its secular variations (envelope in Figure 4, upper panel) is defined by the variable “y”.

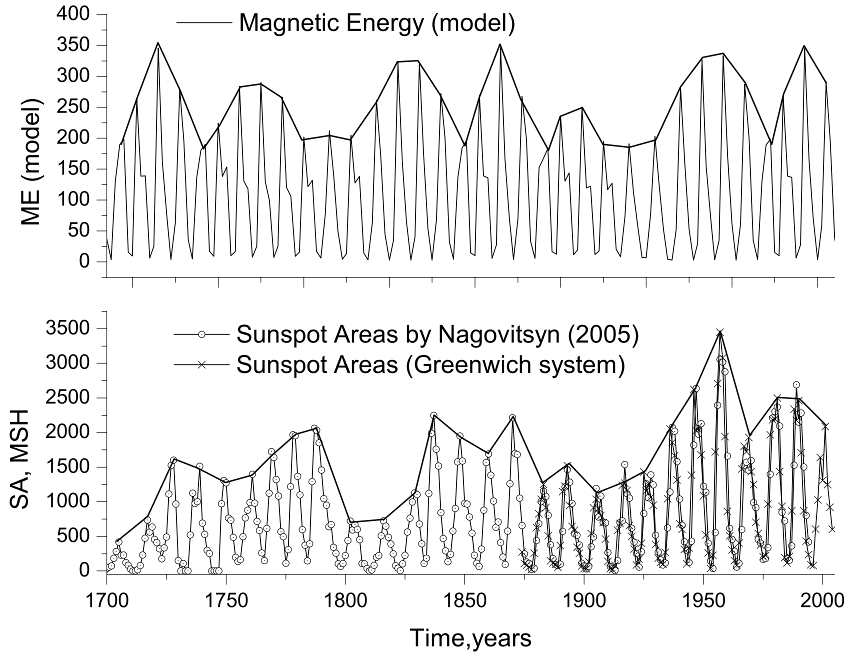

Figure 4. Sunspot area indices and their envelope compared with the model. Upper panel – magnetic energy calculated from the model using the Runge-Kutta fourth-order scheme. Bottom panel - sunspot area indices derived from the combination of sunspot numbers and group sunspot numbers by Nagovitsyn (2005).

The magnetic energy in the model has a secular variation (envelope in Figure 4, upper panel) similar to that of the sunspot area indices derived by Nagovitsyn (2005) from the Wolf sunspot number and the Hoyt-Shatten group sunspot number indices (envelope in Figure 4, bottom panel). This envelope is defined by the variable “y” of the ODE systems (2,3) which proposed to be responsible for the secular torsional oscillations.

The current I_2 behaves similarly to that in Figure 2 and produces regular sign changes (the Hale law of polarity changes). With different parameter leading to deeper modulation of the Schwabe cycle (*e.g.* bigger ε or bigger F_0) we found pairs of Schwabe cycles without a change of polarity during the global minima. It can be seen from the last equation of (3) that such parameter changes lead to the deeper modulation of the main system (1) and during the global minimum epoch it can return the system from periodic behavior to its chaotic regime (Figure 3, from B to A) when the trajectory of the solution can move around each of the focucii instead of moving around both of them. With very small modulation ($\varepsilon \rightarrow 0$), ODEs (1) and (2) become independent in the combined system (3), and the Hale cycle becomes periodic. With very deep modulation (Figure 3 D), the combined system produces a solution with the permanent chaos. So, the proposed "toy" model predicts the possibility of the Hale pair being without polarity change during the deep global minimum. Unfortunately this fact cannot be verified with present day observations.

The quality of the fitting of observable indices by the model is constrained because of the positive Lyapunov exponent in ODEs system (2). The positive exponent leads to the growth of any infinitesimal initial uncertainty and the model cannot fit the observed solar activity indices during intervals longer than the Lyapunov time (210 years).

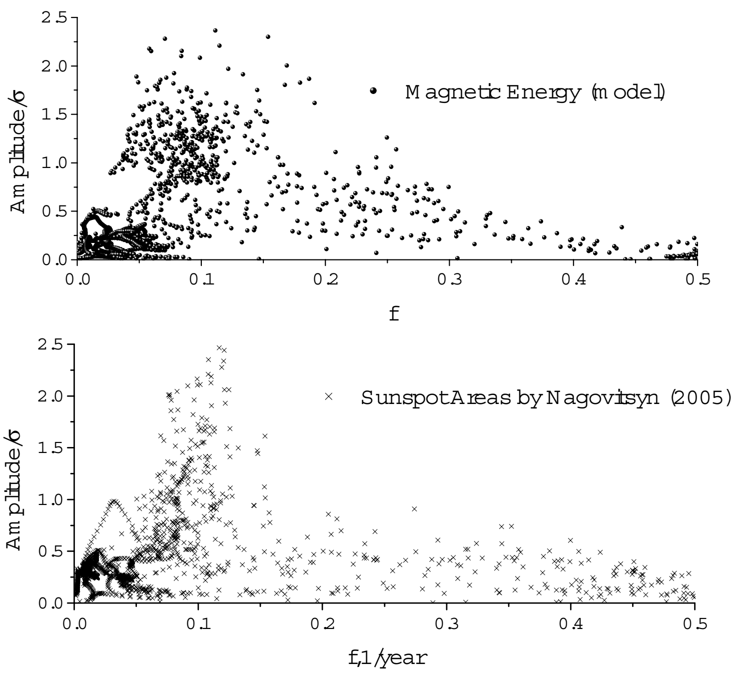

Figure 5. Amplitude-frequency diagram of the sunspot area indices compared with the model. The plot is calculated using Hilbert-Huang transform (Huang *et al.* 1998) of the time series of Figure 4.

Consequently, an invariant of the dynamics should be used to verify the quality of the model. One of the possible invariants is the amplitude–frequency (AF) dependence diagram which was found to be an effective tool for solar activity modeling (Nagovitsyn, 1997). Using the Hilbert-Huang (Huang *et al.*, 1998) spectrum, we calculated the AF diagram for the model magnetic flux and sunspot area indices (Figure 5).

The following features can be seen directly from the sunspot areas AF diagram (Figure 3, bottom panel):

- The AF diagram of the secular solar variations (frequency range *f*=0÷0.05 1/year) has a triangle shape *i.e.* the longer cycle is the bigger one on average, but longer cycles have the broader amplitude spectrum.
- The Schwabe cycle (*f*=0.05÷0.15 1/year) AF diagram describes the larger amplitude of the shorter cycles.
- These two different types of variations in the AF diagram are connected with the narrow bridge which is located at f~0.05.

These three features can also be seen in the AF diagram of the magnetic flux calculated from the model (Figure 5, upper panel).

Conclusions

The "toy" model of the magnetic cycle of solar activity in terms of the ODEs system presented here can describe the evolution of solar activity indices, mainly specific features of the AF diagram of both the 11-year and secular variations of solar activity.

Low-dimensional chaos found in the model of the secular variations of solar activity with the positive Lyapunov exponent and with the predictability horizon of 210 years.

The proposed "toy" model predicts the possibility of the Hale pair being without polarity change during the deep global minima.

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References

Bard, E., Raisbeck, G., Yiou, F., and Jouzel, J.: 2000, *Tellus B*, **52**, 985.

Bullard, E.C.: 1955, *Proc. Camb. Phil. Soc.* **51**, 744.

Durney, B.R.: 2000, *Solar Phys.* **197**, 215.

Ershov S.V., Malinetskii G.G., and Ruzmaikin A.A.: 1989, *Geophys. Astrophys. Fl. Dyn.* **47**, 251.

Hide, R. and Moroz, I. M.: 1999, *Physica D*. **134**, 287.

Hide, R.: 1997, *Phys. Earth Plan. Int.* **103**, 281.

Howard, R. and Labonte, B. J.: 1980, *Astrophys. J.* **239**, L33.

Hoyt, D.V. and Schatten, K. H.: 1996, *Solar Phys.* **165**, 181.

Huang N.E., Shen, Z., Long, S.R., Wu, M.C., Shin, H.H., Zheng, Q., Yen, N.C., Tung, C.C., and Liu H.H.: 1998, Proc. R. Soc. Lond. A **454**, 903.

Kitchatinov, L.L., Pipin, V.V., Makarov, V.I., and Tlatov, A.G.: 1999, *Solar Phys.* **189**, 227.

Knobloch, E. and Landsberg, A. S.: 1996, *Mon. Not. Roy. Astron. Soc.* **278**, 294.

Nagovitsyn Yu.A..: 1997, *Astron. Let.* **23**,742.

Nagovitsyn, Yu.A.: 2005, *Astron. Let.* **31**, 557.

Ogurtsov, M.G., Kocharov, G.E., Lindholm, M., Meriläinen, J., Eronen, M., and Nagovitsyn, Yu. A..: 2002, *Solar Phys*. **205**, 403.

Rikitake, T.: 1958, *Proc. Camb. Phil. Soc.* **54**, 89.

Ruzmaikin, A.A.: 1981, *Comm. Astrophys*. **9**, 85.

Solanki, S.K. and Shüssler, M.: 2005, *Mem. S.A.It*. **76**, 781.

Solanki, S.K., Usoskin, I.G., Kromer, B., Schüssler, M., and Beer, J.: 2004, *Nature*, **431**, 1084.

Tavakol, R.: 1978, *Nature*. **276**, 802.

Volobuev, D.: 2004, *Solar Phys.* **224**, 387.

Wolf, A., Swift, J.B., Swinney, H.L., and Vastano, J. A.: 1985, *Physica D*. **16**, 285.

Yoshimura, H.: 1981, *Astrophys. J.* **247**, 1102.